

${}_3F_2(1)$ hypergeometric function and quadratic R -matrix algebra

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Abstract

We construct a class of representations of the quadratic R -matrix algebra given by the reflection equation with the spectral parameter,

$$R(u-v)T^{(1)}(u)R(u+v)T^{(2)}(v)=T^{(2)}(v)R(u+v)T^{(1)}(u)R(u-v),$$

in terms of certain ordinary difference operators. These operators turn out to act as parameter shifting operators on the ${}_3F_2(1)$ hypergeometric function and its limit cases and on classical orthogonal polynomials. The relationship with the factorisation method will be discussed.

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1 Introduction

Let V be a complex vector space. The quantum R -matrix is a meromorphic operator-valued function $R : \mathbb{C} \rightarrow \text{End}(V \otimes V)$ which satisfies the quantum Yang-Baxter equation of the form [1, 3, 12]

$$R^{(12)}(u-v)R^{(13)}(u-w)R^{(23)}(v-w) = R^{(23)}(v-w)R^{(13)}(u-w)R^{(12)}(u-v).$$

Let us fix the following solution of this equation

$$R(u) = u + \kappa P, \quad u, \kappa \in \mathbb{C}, \quad P(x \otimes y) = y \otimes x, \quad x, y \in V. \quad (1.1)$$

Consider $V = \mathbb{C}^2$ then in the standard basis we have the following 4×4 R -matrix

$$R(u) = \begin{pmatrix} u + \kappa & 0 & 0 & 0 \\ 0 & u & \kappa & 0 \\ 0 & \kappa & u & 0 \\ 0 & 0 & 0 & u + \kappa \end{pmatrix}. \quad (1.2)$$

Introduce a 2×2 matrix

$$U(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad (1.3)$$

with a priori non-commuting entries depending on a so-called *spectral parameter* u which is arbitrary complex. The *QISM II algebra* [5], or the algebra given by the *reflection equation*, is the algebra generated by the matrix elements of $U(u)$ for all u subject to a quadratic relation involving two R -matrices [6, 11]:

$$R(u-v)U^{(1)}(u)R(u+v-\kappa)U^{(2)}(v) = U^{(2)}(v)R(u+v-\kappa)U^{(1)}(u)R(u-v). \quad (1.4)$$

Here we use the notation $U^{(1)}(u) = U(u) \otimes I$, $U^{(2)}(v) = I \otimes U(v)$. The QISM II algebra, or the *U-algebra* is an example of a *quadratic R-matrix algebra*. From now on we assume that $\kappa = 1$. For $\kappa \neq 0$ in (1.2) this means no loss of generality.

In the present article we construct a class of representations of very simple type (U -operators of rank 1) of the U -algebra. We require moreover a certain symmetry property (unitarity) for an U -operator. We will consider U -operators (1.3) for which certain matrix elements will be realized in terms of ordinary difference operators acting as parameter shifting operators on the ${}_3F_2(1)$ generalized hypergeometric function and its limit cases. Specialization then yields shift operator actions on classical orthogonal polynomials (more specifically, on Hahn polynomials which are generic ones for the representations under consideration). We will also point out a close connection of our results with the factorization method for second order difference equations.

Our operators will act on special functions $F(u)$ which appear for each $u \in \mathbb{C}$ as a solution of the equation

$$C(u) F(u) = 0, \quad (1.5)$$

i.e., as functions annihilated by one of the two off-diagonal elements (always chosen to be $C(u)$) of an U -operator. The operators $A(u)$ and $-A(-u)$ then give the shifting of the parameter u by ± 1 , respectively:

$$A(u)F(u) = \Delta_-(u - \tfrac{1}{2})F(u-1), \quad -A(-u)F(u) = \Delta_+(u + \tfrac{1}{2})F(u+1). \quad (1.6)$$

Here the $\Delta_{\pm}(u)$ are certain scalars depending on u which factorize the quantum determinant $\Delta(u)$ of an U -operator:

$$\Delta(u) = \Delta_+(u)\Delta_-(u). \quad (1.7)$$

The quantum determinant for the U -algebra is a certain quadratic expression in the generators with the property that it is the generating function for the center of the algebra. So, in an irreducible representation it is, under suitable assumptions, scalar for each u .

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2 More about the QISM II algebra

We will always assume the following relations (symmetry property when changing the sign of u) for our U -operators:

$$\begin{aligned} -A(-u) &= D(u) - (A(u) + D(u))/(2u + 1), \\ -D(-u) &= A(u) - (A(u) + D(u))/(2u + 1), \\ B(-u) &= B(u), \quad C(-u) = C(u). \end{aligned} \quad (2.1)$$

Note that the second equality is implied by the first. The equations (2.1) can be rephrased as the unitarity property $U^{-1}(-u + 1) \sim U(u)$ ([11]).

The *quantum determinant* for the U -algebra is defined as follows.

$$\begin{aligned} \Delta(u) &= -D(-u + \frac{1}{2})D(u + \frac{1}{2}) - C(u - \frac{1}{2})B(u + \frac{1}{2}) \\ &= -A(-u + \frac{1}{2})A(u + \frac{1}{2}) - B(u - \frac{1}{2})C(u + \frac{1}{2}) \\ &= -D(u + \frac{1}{2})D(-u + \frac{1}{2}) - C(u + \frac{1}{2})B(u - \frac{1}{2}) \\ &= -A(u + \frac{1}{2})A(-u + \frac{1}{2}) - B(u + \frac{1}{2})C(u - \frac{1}{2}). \end{aligned} \quad (2.2)$$

The quantum determinant is the generating function for the center of the algebra [11].

Relation (1.4) can be rewritten in the following extended form as commutators between the algebra generators $A(u)$, $B(u)$, $C(u)$, and $D(u)$.

$$[B, B] = [C, C] = 0, \quad (2.3)$$

$$[A, A] = -(BC - \tilde{B}C)/(u + v), \quad (2.4)$$

$$[D, D] = -(CB - \tilde{C}B)/(u + v), \quad (2.5)$$

$$\begin{aligned} [A, B] &= -(AB - \tilde{A}B)/(u - v) - (AB + BD)/(u + v - 1) \\ &\quad -(AB + BD - \tilde{A}B - \tilde{B}D)/(u - v)/(u + v - 1), \end{aligned} \quad (2.6)$$

$$[B, A] = -(BA - \tilde{B}A)/(u - v) + (\tilde{A}B + \tilde{B}D)/(u + v - 1), \quad (2.7)$$

$$\begin{aligned} [A, C] &= -(CA - \tilde{C}A)/(u - v) + (\tilde{C}A + \tilde{D}C)/(u + v - 1) \\ &\quad -(CA + DC - \tilde{C}A - \tilde{D}C)/(u - v)/(u + v - 1), \end{aligned} \quad (2.8)$$

$$[C, A] = -(AC - \tilde{A}C)/(u - v) - (CA + DC)/(u + v - 1), \quad (2.9)$$

$$[D, B] = -(BD - \tilde{B}D)/(u - v) + (\tilde{A}B + \tilde{B}D)/(u + v - 1)$$

$$-(AB + BD - \tilde{A}\tilde{B} - \tilde{B}\tilde{D})/(u - v)/(u + v - 1), \quad (2.10)$$

$$[B, D] = -(DB - \tilde{D}\tilde{B})/(u - v) - (AB + BD)/(u + v - 1), \quad (2.11)$$

$$\begin{aligned} [D, C] &= -(DC - \tilde{D}\tilde{C})/(u - v) - (CA + DC)/(u + v - 1) \\ &\quad -(CA + DC - \tilde{C}\tilde{A} - \tilde{D}\tilde{C})/(u - v)/(u + v - 1), \end{aligned} \quad (2.12)$$

$$[C, D] = -(CD - \tilde{C}\tilde{D})/(u - v) + (\tilde{C}\tilde{A} + \tilde{D}\tilde{C})/(u + v - 1), \quad (2.13)$$

$$[A, D] = -(CB - \tilde{C}\tilde{B})(u + v + 1)/(u^2 - v^2), \quad (2.14)$$

$$[D, A] = -(BC - \tilde{B}\tilde{C})(u + v + 1)/(u^2 - v^2), \quad (2.15)$$

$$\begin{aligned} [B, C] &= -(DA - \tilde{D}\tilde{A})(u + v - 1)/(u^2 - v^2) \\ &\quad -(AA - \tilde{D}\tilde{D})/(u + v), \end{aligned} \quad (2.16)$$

$$\begin{aligned} [C, B] &= -(AD - \tilde{A}\tilde{D})(u + v - 1)/(u^2 - v^2) \\ &\quad -(DD - \tilde{A}\tilde{A})/(u + v). \end{aligned} \quad (2.17)$$

Here we use for brevity the notations: $[A, B]$ means the commutator $[A(u), B(v)]$, where the first parameter is u and the second one is v ; DA stands for the noncommutative operator product $D(u)A(v)$; and $\tilde{D}A$ signifies $D(v)A(u)$ (where v is the first parameter), and so on.

The following Theorem was proved in [5].

Theorem 2.1 *Let W be a complex vector space on which the U -algebra acts by an algebra representation. Suppose \mathcal{D} is a subset of \mathbb{C} of the form $\{u_0 + m \mid m \in \mathbb{Z}, j_- < m < j_+\}$, where $u_0 \in \mathbb{C}$ and $j_{\pm} = \pm\infty$ or integer, such that*

- (i) $\{w \in W \mid C(u)w = 0\}$ is 1-dimensional for any $u \in \mathcal{D}$,
- (ii) if $u \in \mathcal{D}$, $0 \neq w \in W$ and $C(u)w = 0$ then

$$A(u)w \quad \begin{cases} \neq 0, & u \neq u_0 + j_- + 1, \\ = 0, & u = u_0 + j_- + 1, \end{cases} \quad (2.18)$$

$$-A(-u)w \quad \begin{cases} \neq 0, & u \neq u_0 + j_+ - 1, \\ = 0, & u = u_0 + j_+ - 1. \end{cases} \quad (2.19)$$

For each $u \in \mathcal{D}$ choose $0 \neq F(u) \in W$ such that $C(u)F(u) = 0$. Then

$$\begin{aligned} A(u)F(u) &= \Delta_-(u - \frac{1}{2})F(u - 1), & u \in \mathcal{D}, \quad u \neq u_0 + j_- + 1, \\ -A(-u)F(u) &= \Delta_+(u + \frac{1}{2})F(u + 1), & u \in \mathcal{D}, \quad u \neq u_0 + j_+ - 1, \end{aligned}$$

for certain scalar functions $\Delta_{\pm}(u \pm \frac{1}{2})$. Furthermore, the operator $\Delta(u)$, when acting on $\text{Span}\{F(v) \mid v \in \mathcal{D}\}$, is scalar for $u \in \mathcal{D} \pm \frac{1}{2}$ and it satisfies

$$\Delta(u) = \begin{cases} \Delta_+(u)\Delta_-(u), & u \in \mathcal{D} \pm \frac{1}{2}, \quad u \neq u_0 + j_{\pm} \mp \frac{1}{2}, \\ 0, & u = u_0 + j_{\pm} \mp \frac{1}{2}. \end{cases} \quad (2.20)$$

3 The function ${}_3F_2(1)$

Let us consider the following generalized hypergeometric function:

$$F \equiv {}_3F_2(1) = {}_3F_2(a, b, c; d, e; 1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!},$$

$$\operatorname{Re} (a + b + c - d - e - 1) < -1, \quad d, e \neq 0, -1, -2, \dots,$$

$$(\alpha)_n = \alpha(\alpha + 1) \dots (\alpha + n - 1), \quad (\alpha)_0 = 1.$$

If in this function one, and only one, of the variables is increased or decreased by unity, the resultant function is said to be contiguous to the F above. We will use the notation Δ_α^\pm for the operators defined by the following:

$$\Delta_a^+ F = F(a+) - F, \quad \Delta_d^- F = F(d-) - F.$$

There are 45 relations each expressing F linearly in terms of two of its 10 contiguous functions (see, for instance, [10]). In what follows we will write down a full list of those 45 relations using the operators Δ_α^\pm . We are giving operator's equalities but it is implied that they are true only when acting on the F .

3.1 Contiguous function relations

The first 10 simple relations have the form

$$\alpha \Delta_\alpha^+ = \beta \Delta_\beta^+, \quad \alpha, \beta \in \{a, b, c\}, \quad (3.1)$$

$$\alpha \Delta_\alpha^+ = (\delta - 1) \Delta_\delta^-, \quad \alpha \in \{a, b, c\}, \quad \delta \in \{d, e\}, \quad (3.2)$$

$$(d - 1) \Delta_d^- = (e - 1) \Delta_e^-. \quad (3.3)$$

The next 10 relations have the following form. The triple (α, β, γ) in the formulas below means a cyclic permutation of (a, b, c) while pair (δ, ε) means that of (d, e) .

$$(\alpha - d)(\alpha - e) \Delta_\alpha^- - \alpha \gamma = (\beta - d)(\beta - e) \Delta_\beta^- - \beta \gamma, \quad (3.4)$$

$$(\alpha - \varepsilon) \Delta_\alpha^- = \frac{(\beta - \delta)(\gamma - \delta)}{\delta} \Delta_\delta^+ + \frac{\beta \gamma}{\delta}, \quad (3.5)$$

$$\frac{(a-d)(b-d)(c-d)}{d} \Delta_d^+ + \frac{abc}{d} = \frac{(a-e)(b-e)(c-e)}{e} \Delta_e^+ + \frac{abc}{e}. \quad (3.6)$$

One can supply 25 more equalities using the above 20. In the next 6 formulas the triple (α, β, γ) means any permutation of (a, b, c) .

$$(\alpha - d)(\alpha - e) \Delta_\alpha^- + \beta(a + b + c + 1 - d - e) \Delta_\beta^+ + \beta \gamma = 0. \quad (3.7)$$

In the next 6 formulas the triple (α, β, γ) means a cyclic permutation of (a, b, c) and $\delta \in \{d, e\}$.

$$(\alpha - d)(\alpha - e) \Delta_\alpha^- + (\delta - 1)(a + b + c + 1 - d - e) \Delta_\delta^- + \beta \gamma = 0. \quad (3.8)$$

In the next 6 formulas $\alpha \in \{a, b, c\}$ and $\delta \in \{d, e\}$.

$$\frac{(a-\delta)(b-\delta)(c-\delta)}{\delta} \Delta_\delta^+ + \frac{abc}{\delta} + \alpha(a + b + c + 1 - d - e) \Delta_\alpha^+ = 0. \quad (3.9)$$

In the next 2 formulas (δ, ε) is a permutation of (d, e) .

$$\frac{(a-\delta)(b-\delta)(c-\delta)}{\delta} \Delta_\delta^+ + \frac{abc}{\delta} + (\varepsilon - 1)(a + b + c + 1 - d - e) \Delta_\varepsilon^- = 0. \quad (3.10)$$

We conclude by 5 important relations. In first three of them (α, β, γ) is a cyclic permutation of (a, b, c) .

$$(\alpha - d)(\alpha - e) \Delta_\alpha^- + \alpha(a + b + c + 1 - d - e) \Delta_\alpha^+ + \beta \gamma = 0. \quad (3.11)$$

In the last two equalities $\delta \in \{d, e\}$.

$$\frac{(a-\delta)(b-\delta)(c-\delta)}{\delta} \Delta_\delta^+ + \frac{abc}{\delta} + (\delta - 1)(a + b + c + 1 - d - e) \Delta_\delta^- = 0. \quad (3.12)$$

The last 5 relations have the following interpretation: they are second order difference equations for the function F considered as a function of the corresponding variable (a, b, c, d or e).

The formulas for the Charlier, Krawtchouk, Meixner, and Hahn polynomials can be obtained through the corresponding specifications of the parameters a, b, c, d , and e .

Combining some of the above relations (3.1)–(3.10) one can get the following *operator shift actions* (the action of the first order difference operators w.r.t. the variable c is equivalent to the shifting of the parameter u):

$$\begin{aligned} & \left[\frac{1}{2}(u - \frac{1}{2})^2 + (-c + \frac{1}{4} + \frac{d+e-a}{2})(u - \frac{1}{2}) - \frac{1}{2}(a + \frac{1}{2})^2 + \frac{1}{2}(-de + d + e) \right. \\ & \left. + c(a - \frac{1}{2}) - (c - d)(c - e)\Delta_c^- + \frac{\delta}{u - \frac{1}{2}} \right] {}_3F_2(a + u, a - u, c; d, e; 1) \\ &= \frac{(a-u)(a-d+u)(a-e+u)}{2u-1} {}_3F_2(a + u - 1, a - u + 1, c; d, e; 1), \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \left[-\frac{1}{2}(u + \frac{1}{2})^2 + (-c + \frac{1}{4} + \frac{d+e-a}{2})(u + \frac{1}{2}) + \frac{1}{2}(a + \frac{1}{2})^2 - \frac{1}{2}(-de + d + e) \right. \\ & \left. - c(a - \frac{1}{2}) + (c - d)(c - e)\Delta_c^- + \frac{\delta}{u + \frac{1}{2}} \right] {}_3F_2(a + u, a - u, c; d, e; 1) \\ &= \frac{(a+u)(a-d-u)(a-e-u)}{2u+1} {}_3F_2(a + u + 1, a - u - 1, c; d, e; 1), \end{aligned} \quad (3.14)$$

$$\delta = \frac{1}{2}(a - \frac{1}{2})((a + \frac{1}{2})(a + \frac{1}{2} - d - e) + de).$$

The analogous pair is with c shifted in an opposite direction:

$$\begin{aligned} & \left[-\frac{1}{2}(u - \frac{1}{2})^2 + (-c - \frac{3}{4} + \frac{d+e-a}{2})(u - \frac{1}{2}) + \frac{1}{2}(a - \frac{1}{2})^2 - \frac{1}{2}(de - d - e + 1) \right. \\ & \left. + c(a - \frac{1}{2}) + c(c + 2a - d - e + 1)\Delta_c^+ + \frac{\delta}{u - \frac{1}{2}} \right] {}_3F_2(a + u, a - u, c; d, e; 1) \\ &= \frac{(a-u)(a-d+u)(a-e+u)}{2u-1} {}_3F_2(a + u - 1, a - u + 1, c; d, e; 1), \end{aligned} \quad (3.15)$$

$$\begin{aligned} & \left[\frac{1}{2}(u + \frac{1}{2})^2 + (-c - \frac{3}{4} + \frac{d+e-a}{2})(u + \frac{1}{2}) - \frac{1}{2}(a - \frac{1}{2})^2 + \frac{1}{2}(de - d - e + 1) \right. \\ & \left. - c(a - \frac{1}{2}) - c(c + 2a - d - e + 1)\Delta_c^+ + \frac{\delta}{u + \frac{1}{2}} \right] {}_3F_2(a + u, a - u, c; d, e; 1) \\ &= \frac{(a+u)(a-d-u)(a-e-u)}{2u+1} {}_3F_2(a + u + 1, a - u - 1, c; d, e; 1), \end{aligned} \quad (3.16)$$

$$\delta = \frac{1}{16}(2a - 1)(2a + 1 - 2e)(2a + 1 - 2d).$$

4 Rank 1 quadratic algebra

The (unitary) U -algebra is the algebra with the matrix elements of $U(u)$ as generators and with the quadratic relations given in the form of the reflection equation (1.4)

([5, 6, 11]). We consider here the reflection equation with the spectral parameter, with 2-dimensional auxiliary space and with simplest R -matrix of the XXX -type (1.1).

Let us pass to a quotient algebra by adding relations stating that $(u - \frac{1}{2})U(u)$ is a polynomial of degree ≤ 3 in u . In other words, we make the ansatz that $U(u)$ is of the form

$$\begin{aligned} U(u) &= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \\ &= (u - \frac{1}{2})^2 U_2 + (u - \frac{1}{2}) U_1 + U_0 + (u - \frac{1}{2})^{-1} U_{-1}, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} U_2 &= \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix}, \quad U_1 = \begin{pmatrix} A_1 & B_2 \\ C_2 & A_1 - 2A_2 \end{pmatrix}, \\ U_0 &= \begin{pmatrix} A_0 & B_0 + \frac{1}{4}B_2 \\ C_0 + \frac{1}{4}C_2 & -A_0 + 2A_1 - 2A_2 \end{pmatrix}, \quad U_{-1} = \begin{pmatrix} A_{-1} & 0 \\ 0 & A_{-1} \end{pmatrix}. \end{aligned} \quad (4.2)$$

Then we get the algebra with the matrix elements of the U_i ($i = 2, 1, 0, -1$) as generators and with relations

$$U_2^{(1)} U_{2,1,0,-1}^{(2)} = U_{2,1,0,-1}^{(2)} U_2^{(1)}, \quad U_{-1}^{(1)} U_{2,1,0,-1}^{(2)} = U_{2,1,0,-1}^{(2)} U_{-1}^{(1)}, \quad (4.3)$$

$$[U_1^{(1)}, U_0^{(2)}] = -[P, U_2^{(1)} U_0^{(2)}] - U_2^{(1)} P U_0^{(2)} + U_0^{(2)} P U_2^{(1)}, \quad (4.4)$$

$$[U_0^{(1)}, U_0^{(2)}] = -[\{P, U_1^{(1)}\}, U_0^{(2)}] - 2A_{-1}[P, U_2^{(1)}] + [U_0, U_2]^{(2)}. \quad (4.5)$$

Here curved brackets mean anticommutator. The relations (4.3) imply that the entries of the U_2 and U_{-1} matrices are in the center of the algebra. Let us pass once more to a quotient algebra by adding the relations

$$U_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \quad A_{-1} = \delta, \quad (4.6)$$

for certain $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. We thus obtain an algebra with generators A_0, A_1, B_0, C_0 and relations

$$\begin{aligned} [A_1, A_0] &= \gamma B_0 - \beta C_0, \\ [A_1, B_0] &= -2\alpha B_0 + \beta \left(2A_0 - 2A_1 + \frac{3}{2}\alpha\right), \\ [A_1, C_0] &= 2\alpha C_0 + \gamma \left(-2A_0 + 2A_1 - \frac{3}{2}\alpha\right), \\ [A_0, B_0] &= -\{A_1, B_0\} + \beta \left(2A_0 - \frac{5}{2}A_1 + 2\alpha + 2\delta\right), \\ [A_0, C_0] &= \{A_1, C_0\} + \gamma \left(-2A_0 + \frac{5}{2}A_1 - 2\alpha - 2\delta\right), \\ [B_0, C_0] &= -2\{A_0, A_1\} + 4(A_1 - \alpha)^2 + 4\alpha A_0 + 4\alpha\delta. \end{aligned} \quad (4.7)$$

The quantum determinant has the form

$$\Delta(u) = -(\alpha^2 + \beta\gamma)u^4 + Q_2 u^2 + Q_0 + \delta^2 u^{-2}, \quad (4.8)$$

$$Q_2 = A_1^2 - 2\alpha A_0 - \gamma B_0 - \beta C_0 + \frac{1}{2}\beta\gamma, \quad (4.9)$$

$$Q_0 = -A_0^2 - B_0 C_0 + 2\delta A_1 - \frac{1}{4}\gamma B_0 - \frac{1}{4}\beta C_0 - \frac{1}{16}\beta\gamma. \quad (4.10)$$

Here the right hand sides of (4.9) and (4.10) give operators in the center of the algebra. We consider (4.9) and (4.10) as added relations, for a certain choice of $Q_2, Q_0 \in \mathbb{C}$. So

a representation of the algebra with relations (4.3)–(4.5) which has the property that all elements in the center of the algebra are represented as scalars, can also be viewed as a representation of the algebra with relations (4.7), (4.9) and (4.10) for a certain choice of $\alpha, \beta, \gamma, \delta, Q_0, Q_2$.

Let us change the generators A_1, A_0, B_0, C_0 to the new ones $\tilde{A}_1, \tilde{A}_0, \tilde{B}_0, \tilde{C}_0$:

$$\tilde{A}_1 = A_1 - \alpha, \quad \tilde{A}_0 = A_0 - A_1 + \alpha, \quad \tilde{B}_0 = B_0 + \frac{\beta}{4}, \quad \tilde{C}_0 = C_0 + \frac{\gamma}{4}.$$

The relations for new generators can be rewritten in the following form:

$$\begin{aligned}\tilde{A}_0\tilde{A}_1 &= \tilde{A}_1\tilde{A}_0 - \gamma\tilde{B}_0 + \beta\tilde{C}_0, \\ \tilde{B}_0\tilde{A}_1 &= \tilde{A}_1\tilde{B}_0 + 2\alpha\tilde{B}_0 - 2\beta\tilde{A}_0, \\ \tilde{C}_0\tilde{A}_1 &= \tilde{A}_1\tilde{C}_0 - 2\alpha\tilde{C}_0 + 2\gamma\tilde{A}_0, \\ \tilde{B}_0\tilde{A}_0 &= \tilde{A}_0\tilde{B}_0 + 2\tilde{A}_1\tilde{B}_0 + 2\alpha\tilde{B}_0 - 2\beta\tilde{A}_0 - 2\beta\delta, \\ \tilde{C}_0\tilde{A}_0 &= \tilde{A}_0\tilde{C}_0 - 2\tilde{A}_1\tilde{C}_0 + 2\alpha\tilde{C}_0 - 2\gamma\tilde{A}_0 + 2\gamma\delta, \\ \tilde{C}_0\tilde{B}_0 &= \tilde{B}_0\tilde{C}_0 + 4\tilde{A}_1\tilde{A}_0 - 2\gamma\tilde{B}_0 + 2\beta\tilde{C}_0 - 4\alpha\delta.\end{aligned}\tag{4.11}$$

Let we denote by the $\tilde{\mathcal{A}}$ the algebra given by the generators $\tilde{A}_1, \tilde{A}_0, \tilde{B}_0, \tilde{C}_0$ and the relations (4.11). We have the following

Theorem 4.1 (*Poincare-Birkhoff-Witt property*) *Let $\tilde{\mathcal{A}}^{(n)} \subset \tilde{\mathcal{A}}$ be the linear span of monomials of degree n on generators $\tilde{A}_1, \tilde{A}_0, \tilde{B}_0, \tilde{C}_0$. Then the dimension of $\tilde{\mathcal{A}}^{(n)}$ is equal to the dimension of the space of monomials of degree n on 4 commuting variables $\tilde{A}_1, \tilde{A}_0, \tilde{B}_0, \tilde{C}_0$.*

Proof The proof can be done using the Diamond Lemma [2]. The defining relations (4.11) respect the following ordering $\tilde{A}_1 < \tilde{A}_0 < \tilde{B}_0 < \tilde{C}_0$. It can be simply verified that there are no additional relations appearing in the following ambiguities:

$$\tilde{C}_0\tilde{B}_0\tilde{A}_0, \quad \tilde{C}_0\tilde{B}_0\tilde{A}_1, \quad \tilde{B}_0\tilde{A}_0\tilde{A}_1, \quad \tilde{C}_0\tilde{A}_0\tilde{A}_1.$$

The group $\text{GL}(2, \mathbb{C})$ acts naturally on the space of U -operators

$$\tilde{U}(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} U(u) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},\tag{4.12}$$

where $ad - bc \neq 0$. We remark that such transformation is compatible to the unitarity property (2.1). One may use this transformation to get a more suitable set of the scalars α, β, γ . It is always possible to arrange that

$$\beta = 0.\tag{4.13}$$

Then there are only three cases to study:

$$\begin{aligned}Case \quad i) : \quad &\gamma = 1, \quad \alpha = \frac{1}{2}; \\ Case \quad ii) : \quad &\gamma = 1, \quad \alpha = 0; \\ Case \quad iii) : \quad &\gamma = \alpha = 0.\end{aligned}\tag{4.14}$$

The case *ii*) has been studied in [5]. Here we are dealing more with the cases *i*) and *iii*).

5 Homomorphisms into $\mathcal{U}(g)$

In this Section we are giving some homomorphisms of the quadratic algebra with the generators A_1, A_0, B_0, C_0 and the relations (4.7) into the universal enveloping algebra $\mathcal{U}(g)$ of the Lie algebra g for all the cases (4.14).

5.1 The case i)

Consider the algebra \mathcal{A} with generators $A_1, A_0, B_0, C_0, \delta$ and with two sets of relations: relations (4.7) with $\alpha = 0, \beta = \gamma = 1$, and relations stating that δ is in the center of the algebra. The case of the scalars α, β, γ just chosen is equivalent to the case i) in (4.14) under a certain transformation of the form (4.12). There is a homomorphism of this algebra into the universal enveloping algebra $\mathcal{U}(\text{o}(4))$ of the Lie algebra $\text{o}(4)$. A Lie group corresponding to $\text{o}(4)$ is the group of rotations of 4-dimensional Euclidean space. The Lie algebra $\text{o}(4)$ is 6-dimensional. It can be described by the generators $\mathcal{P}^\pm, \mathcal{P}^3, \mathcal{J}^\pm, \mathcal{J}^3$ and commutation relations

$$\begin{aligned} [\mathcal{J}^3, \mathcal{J}^\pm] &= \pm \mathcal{J}^\pm, & [\mathcal{J}^3, \mathcal{P}^\pm] &= [\mathcal{P}^3, \mathcal{J}^\pm] = \pm \mathcal{P}^\pm, \\ [\mathcal{J}^+, \mathcal{P}^+] &= [\mathcal{J}^-, \mathcal{P}^-] = [\mathcal{J}^3, \mathcal{P}^3] = 0, \\ [\mathcal{J}^+, \mathcal{J}^-] &= 2\mathcal{J}^3, & [\mathcal{J}^+, \mathcal{P}^-] &= [\mathcal{P}^+, \mathcal{J}^-] = 2\mathcal{P}^3, \\ [\mathcal{P}^3, \mathcal{P}^\pm] &= \pm \mathcal{J}^\pm, & [\mathcal{P}^+, \mathcal{P}^-] &= 2\mathcal{J}^3. \end{aligned}$$

The center of the universal enveloping algebra is generated by two Casimir elements:

$$C = (\mathcal{P}^3)^2 + \frac{1}{2}\{\mathcal{P}^+, \mathcal{P}^-\} + (\mathcal{J}^3)^2 + \frac{1}{2}\{\mathcal{J}^+, \mathcal{J}^-\}, \quad \tilde{C} = \frac{1}{2}(\mathcal{P}^+ \mathcal{J}^- - \mathcal{P}^- \mathcal{J}^+) + \mathcal{P}^3 \mathcal{J}^3.$$

It is now straightforward to verify that the relations for the generators of \mathcal{A} are satisfied when we put these generators equal to the following elements of $\mathcal{U}(\text{o}(4))$.

$$\begin{aligned} A_1 &= \mathcal{P}^3, & A_0 &= \frac{1}{2}(\mathcal{P}^+ \mathcal{J}^- - \mathcal{P}^- \mathcal{J}^+), & B_0 &= -(\mathcal{J}^3)^2 - \frac{1}{2}\{\mathcal{P}^+, \mathcal{P}^-\} - \frac{1}{4}, \\ C_0 &= -(\mathcal{J}^3)^2 - \frac{1}{2}\{\mathcal{J}^+, \mathcal{J}^-\} - \frac{1}{4}, & \delta &= -\tilde{C} \mathcal{J}^3. \end{aligned}$$

5.2 The case ii)

The case ii) ($\alpha = \beta = 0, \gamma = 1$) corresponds to the contraction of the algebra $\text{o}(4)$ to the Lie algebra $\text{e}(3)$. So, there is a homomorphism of such algebra into the universal enveloping algebra $\mathcal{U}(\text{e}(3))$. A Lie group corresponding to $\text{e}(3)$ is the group of motions of 3-dimensional Euclidean space. The Lie algebra $\text{e}(3)$ is 6-dimensional. It can be described by the generators $\mathcal{P}^\pm, \mathcal{P}^3, \mathcal{J}^\pm, \mathcal{J}^3$ and commutation relations

$$\begin{aligned} [\mathcal{J}^3, \mathcal{J}^\pm] &= \pm \mathcal{J}^\pm, & [\mathcal{J}^3, \mathcal{P}^\pm] &= [\mathcal{P}^3, \mathcal{J}^\pm] = \pm \mathcal{P}^\pm, \\ [\mathcal{J}^+, \mathcal{P}^+] &= [\mathcal{J}^-, \mathcal{P}^-] = [\mathcal{J}^3, \mathcal{P}^3] = 0, \\ [\mathcal{J}^+, \mathcal{J}^-] &= 2\mathcal{J}^3, & [\mathcal{J}^+, \mathcal{P}^-] &= [\mathcal{P}^+, \mathcal{J}^-] = 2\mathcal{P}^3, \\ [\mathcal{P}^3, \mathcal{P}^\pm] &= [\mathcal{P}^+, \mathcal{P}^-] = 0. \end{aligned}$$

The center of the universal enveloping algebra is generated by two Casimir elements:

$$C = (\mathcal{P}^3)^2 + \mathcal{P}^+ \mathcal{P}^-, \quad \tilde{C} = \frac{1}{2} (\mathcal{P}^+ \mathcal{J}^- + \mathcal{P}^- \mathcal{J}^+) + \mathcal{P}^3 \mathcal{J}^3.$$

The explicit homomorphism has the form [5]

$$\begin{aligned} A_1 &= \mathcal{P}^3, & A_0 &= \frac{1}{2} (\mathcal{P}^+ \mathcal{J}^- - \mathcal{P}^- \mathcal{J}^+), & B_0 &= -\mathcal{P}^+ \mathcal{P}^-, \\ C_0 &= -(\mathcal{J}^3)^2 - \frac{1}{2} \{\mathcal{J}^+, \mathcal{J}^-\} - \frac{1}{4}, & \delta &= -\tilde{C} \mathcal{J}^3. \end{aligned}$$

5.3 The case iii)

In the case iii) ($\alpha = \beta = \gamma = 0$) the generator A_1 is in the center of the algebra and we put $A_1 = -\frac{1}{2}$. The commutation relations (4.7) become linear and we have the following homomorphism of such algebra to the Lie algebra $\text{o}(3)$:

$$A_0 = \mathcal{J}^3 - \frac{1}{2}, \quad B_0 = \mathcal{J}^+, \quad C_0 = \mathcal{J}^-, \quad \delta \in \mathbb{C},$$

where the $\text{o}(3)$ generators $\mathcal{J}^\pm, \mathcal{J}^3$ satisfy the commutation relations

$$[\mathcal{J}^3, \mathcal{J}^\pm] = \pm \mathcal{J}^\pm, \quad [\mathcal{J}^+, \mathcal{J}^-] = 2\mathcal{J}^3.$$

6 Realisation via difference operators

Let us consider the generic case i) ($\beta = 0, \alpha = \frac{1}{2}, \gamma = 1$). The following lemma is a slightly extended form of the Lemma 6.2 in [5]. It shows that equations (4.7), (4.9) and (4.10), with (4.13) (and with (4.14) for the case i)), and under the assumption that B_0 is injective, can be equivalently written in a much more simple form.

Lemma 6.1 *Let δ, Q_0, Q_2 be scalars. Let A_1, A_0, B_0, C_0 be operators acting on some linear space. Let B_0 be injective. Then the following three statements are equivalent:*

$$(a) \quad \begin{pmatrix} \frac{1}{2}(u - \frac{1}{2})^2 + (u - \frac{1}{2})A_1 + A_0 + \frac{\delta}{u - \frac{1}{2}} & B_0 \\ u^2 + C_0 & -\frac{1}{2}(u + \frac{1}{2})^2 + (u + \frac{3}{2})A_1 \\ & -A_0 - \frac{1}{2} + \frac{\delta}{u - \frac{1}{2}} \end{pmatrix}$$

is a representation of the U -algebra with quantum determinant $\Delta(u) = -\frac{1}{4}u^4 + Q_2u^2 + Q_0 + \delta^2u^{-2}$;

(b) The six commutators (4.7) and formulas (4.9), (4.10) are valid with $\alpha = \frac{1}{2}, \beta = 0$ and $\gamma = 1$;

(c) The following three equations are valid:

$$[A_1, A_0] = A_1^2 - A_0 - Q_2, \quad (6.1)$$

$$B_0 = A_1^2 - A_0 - Q_2, \quad (6.2)$$

$$B_0 C_0 = 2\delta A_1 - A_0^2 - \frac{1}{4}B_0 - Q_0. \quad (6.3)$$

We now make the restrictive assumption that A_0 is a second order difference operator and A_1 is a scalar function of x :

$$A_0 = A_0^-(x) (1 + \Delta_x^-) + A_0^+(x) (1 + \Delta_x^+) + A_{00}(x), \quad A_1 = A_{10}(x). \quad (6.4)$$

The following approach should now be followed. Find all operators A_0, A_1 of the form (6.4) such that (6.1) is satisfied for some number Q_2 . (It is sufficient to find one solution in each equivalence class formed by gauge transformations.) Then define B_0 by (6.2) and try to define C_0 by (6.3). Fix some function space W on which these operators act. Then the equivalent conditions of Lemma 6.1 are satisfied. Finally check if the conditions of Theorem 2.1 are satisfied for some choice of \mathcal{D} .

Below we are giving two lemmas which can be proved by straightforward computation.

Lemma 6.2 *If we assume (6.4) for the A_0 and A_1 then (6.1) holds if and only if*

$$A_{00}(x) = A_{10}^2(x) - Q_2, \quad (6.5)$$

$$A_0^-(x) (A_{10}(x) - A_{10}(x-1) + 1) = 0, \quad (6.6)$$

$$A_0^+(x) (A_{10}(x) - A_{10}(x+1) + 1) = 0. \quad (6.7)$$

Lemma 6.3 *Let two functions $A_0^\pm(x)$ be not both identically equal to zero. Then there are only two solutions of the equations (6.5)–(6.7):*

- (a) $A_0^+(x) \equiv 0, \quad A_{10}(x) = -x + c_1, \quad A_{00}(x) = (x - c_1)^2 - Q_2,$
- (b) $A_0^-(x) \equiv 0, \quad A_{10}(x) = x + c_1, \quad A_{00}(x) = (x + c_1)^2 - Q_2$

where c_1 is an arbitrary constant and one has an arbitrary function ($A_0^-(x)$ in the case (a) and $A_0^+(x)$ in the case (b)) which can be fixed by the gauge transformation.

The case (a) of the Lemma 6.3 is equivalent (up to a gauge transformation and a constant shift of the independent variable x) to the following realisation of the algebra in terms of the difference operators:

$$\begin{aligned} A_1 &= -x + \frac{1}{4} + \frac{d+e-a}{2}, \\ A_0 &= -\frac{1}{2}(a + \frac{1}{2})^2 + \frac{1}{2}(-de + d + e) + x(a - \frac{1}{2}) - (x - d)(x - e)\Delta_x^-, \\ B_0 &= (x - d)(x - e)(1 + \Delta_x^-), \\ C_0 &= -(x - d)(x - e)\Delta_x^- - x(x + 2a + 1 - d - e)\Delta_x^+ - a^2, \\ Q_2 &= \frac{a}{4} - \frac{d}{4} - \frac{e}{4} + \frac{3a^2}{4} - \frac{ad}{2} - \frac{ae}{2} + \frac{d^2}{4} + \frac{e^2}{4} + 3/16, \\ Q_0 &= -\frac{ed}{4} - \frac{a}{8} + \frac{d}{8} + \frac{e}{8} - eda - \frac{a^2}{8} + \frac{ad}{4} + \frac{ae}{4} - \frac{d^2}{8} \\ &\quad - \frac{e^2}{8} + \frac{ed^2}{4} + \frac{e^2d}{4} + \frac{ed^2a}{2} + \frac{e^2da}{2} - \frac{3}{64} - \frac{3a^4}{4} \\ &\quad + \frac{a^2d}{2} + \frac{a^2e}{2} - \frac{a^3}{2} + a^3d + a^3e - \frac{e^2d^2}{4} - \frac{a^2d^2}{2} - \frac{a^2e^2}{2} - a^2ed, \\ \delta &= \frac{1}{2}(a - \frac{1}{2})((a + \frac{1}{2})(a + \frac{1}{2} - d - e) + de). \end{aligned}$$

The case (b) of the Lemma 6.3 is equivalent (up to a gauge transformation and a constant shift of the independent variable x) to the following realisation of the algebra in terms of the difference operators:

$$\begin{aligned}
A_1 &= x + \frac{3}{4} - \frac{d+e-a}{2}, \\
A_0 &= -\frac{1}{2}(a - \frac{1}{2})^2 + \frac{1}{2}(de - d - e + 1) - x(a - \frac{1}{2}) - x(x + 2a + 1 - d - e)\Delta_x^+, \\
B_0 &= x(x + 2a + 1 - d - e)(1 + \Delta_x^+), \\
C_0 &= -(x - d)(x - e)\Delta_x^- - x(x + 2a + 1 - d - e)\Delta_x^+ - a^2, \\
Q_2 &= \frac{a}{4} - \frac{e}{4} - \frac{d}{4} + \frac{3a^2}{4} + \frac{e^2}{4} - \frac{ea}{2} + \frac{d^2}{4} + 3/16 - \frac{da}{2}, \\
Q_0 &= -\frac{de}{4} - \frac{a}{8} + \frac{e}{8} + \frac{d}{8} - \frac{a^2}{8} - dea - \frac{e^2}{8} + \frac{ea}{4} + \frac{da}{4} - \frac{d^2}{8} \\
&\quad + \frac{de^2}{4} + \frac{d^2e}{4} - a^2de + \frac{de^2a}{2} + \frac{d^2ea}{2} - \frac{3a^4}{4} - \frac{3}{64} + a^3e + a^3d \\
&\quad - \frac{a^2d^2}{2} - \frac{a^2e^2}{2} - \frac{d^2e^2}{4} + \frac{a^2e}{2} + \frac{a^2d}{2} - \frac{a^3}{2}, \\
\delta &= \frac{1}{16}(2a - 1)(2a + 1 - 2e)(2a + 1 - 2d).
\end{aligned}$$

Here the a, d , and e are arbitrary parameters.

These operators act on the functions ${}_3F_2(a + u, a - u, x; d, e; 1)$ (cf. (3.13)–(3.14) and (3.15)–(3.16)).

7 Concluding remarks

It follows from (2.2) (second and fourth formula) that

$$-A(-u + 1)A(u) = B(u - 1)C(u) + \Delta(u - \frac{1}{2}), \quad (7.1)$$

$$-A(u + 1)A(-u) = B(u + 1)C(u) + \Delta(u + \frac{1}{2}). \quad (7.2)$$

So assume that we have a representation of the QISM II algebra on a space of functions in one variable, analytic on a certain region, such that (i) $B(u) = B_0$ is independent of u , (ii) the unitarity property (2.1) is satisfied, (iii) $\Delta(u)$ is scalar for all u , (iv) $A(u)$ is a first order difference operator. Then equations (7.1)–(7.2) show that the second order difference operator $B_0C(u)$ has a suitable form for the factorization method, which was originated by Schrödinger and was due in its definitive form to Infeld and Hull [4].

The factorization method for the second order differential operators under some special assumptions on the type of factorizing operators has been summarized in Miller [7, Ch. 7]. These factorizing operators give rise to a Lie algebra of first order differential operators in two variables or to some operators in the universal enveloping algebra of certain Lie algebras. See [5] as for the R -matrix interpretation. The generalization to the difference operators and some further generalizations have been given in [8, 9].

In this work we have established a connection of the QISM II algebra representations to some characterizing properties of the generalized hypergeometric function ${}_3F_2(1)$. Namely, we have given the algebraic interpretation of the three relations between ${}_3F_2(1)$ and some other two functions which are contiguous to it. On the Hahn polynomials level these formulas are equivalent to the recurrence relation, difference “differentiation” formula and the second order difference equation for the polynomials (operators A_1, A_0 , and C_0).

The presented homomorphisms of the QISM II algebra into $\mathcal{U}(g)$ will allow us to construct some new integrable systems which will be published elsewhere.

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